A Study of Some Lacunary Boundary Value Problems by B-Spline

A Thesis

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(Numerical Analysis)

By

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بِسمِ اللّهِ الرَّحمَنِ الرَّحيمِ

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We certify that we have read this thesis entitled **"A Study of Some Lacunary Boundary Value Problems by B-Spline"** prepared by **(Bzhar Jamal Aziz)**, as the examining committee, examined the student in its content and in what is connected with it, and in our opinion it meets the basic requirements toward the degree of Master of Science in Mathematics (Numerical Analysis)

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Dedications

With affection, this thesis is dedicated to

- My Mother
- My Father
- My brothers and my sister
- Those who taught me and encouraged me to achieve this work

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List of Symbols

The following symbols are used throughout the thesis

Symbols	**Description**
B-Spline	Basis Spline Function
$B_{i,k}(x)$	B-Spline of degree k
$S(x)$	Sum of B-Splines which is the approximate solution
$\|A\|_\infty$	Infinity norm of A
$C^4[a,b]$	The set of all function which is continuous itself and its first three derivatives on the closed interval [a,b]
$C^6[a,b]$	The set of all function which is continuous itself and its first five derivatives on the closed interval [a,b]
B.V.P	Boundary Value Problem

Abstract

The study tries to find a suitable B-Spline $S(x)$ interpolating the lacunary data given on a function and some approximate boundary conditions on the function.

First, introduce Interpolation using B-Splines, types of B-Spline, its degrees and properties of it and then discuss some certain cases of lacunary interpolation using B-Splines were discussed.

In the second chapter, a fourth degree B-Spline has been constructed that is an approximate solution to a function with very limited given lacunary data and approximate boundary conditions, then the error bound for the B-spline is found. Also used to solve boundary value problem.

In the third chapter, a sixth degree B-Spline is formed so as to be an approximate solution of a boundary value problem with limited lacunary interpolation condition. And the type of the approximate boundary condition is different than the one of chapter two.

The boundary conditions given on the function $f(x)$ are approximate boundary conditions and the data given on $f(x)$ is lacunary and limited.

Coefficients of the B-Splines could be found through forming some equations from the lacunary interpolation followed by presenting more equations through the approximate boundary conditions, then to obtain the number of equations to be equal to the number of unknowns.

In order to have a quick and precise result, MATLAB is used to find out the solution of the square system which will be in matrix form, MATLAB is confirming whether the system has a unique solution or not.

CONTENTS

Chapter Three: Conclusion and Future Works

Chapter One

The Concept of Interpolation, Spline and B-Spline

1.1 Introduction

The theory of spline function is a very attractive field of approximation. Usually a spline function is a piecewise polynomial function of degree k in a variable x and it is defined on a region. The places where the pieces meet are known as knots. The number of knots must be equal to, or greater than $k+2$. Thus the spline function has limited support [1].

Spline or piecewise Interpolations are widely used in the method of piecewise polynomial approximation to represent a function that is not analytic. Although in piecewise interpolation the maximum error between a function and its interpolant can be controlled by mesh spacing, but such functions have corners at the joints of two pieces and therefore more data is required than higher order method to get the desired accuracy. Thus for a smooth and more efficient approximation one has to go to piecewise polynomial approximation with higher degree pieces [14].

Higher degree splines are popular for best approximation [3], Rana and Dubey [14] generalized the result of Howell and Varma [8] and obtained best error bounds for quartic spline interpolation. When it comes to aspects of cubic, quartic and spline of degree six, reference may be given to Meir and Sharma [13], Hall and Meyer [7], Gemling - Meyling [6], Dubey [4].

The interest in spline functions is due to the fact that ,spline functions are a good tool for the numerical approximation of functions on the one hand and that they suggest new, challenging and rewarding problems on the other. Piecewise linear functions, as well as step functions ,have long been an important theoretical and practical tools for approximation of functions as said by Jwamer [9] .

In the present thesis, B-spline is used to describe Numerical solution of mathematical problems by strategically researching the existing B-spline techniques.

Basis functions are fast in computation, flexible, differentiable and constrained as required such as periodicity, positivity, … etc.

Some of the commonly used basis functions are powers, Fourier series, spline functions and B-Splines.

B-Splines were investigated as early as the nineteenth century by Nikolai Lobachevsky. The term "B-spline" was coined by Isaac Jacob Schoenberg and is short for basis spline.

It is the first time in numerical analysis, the approximate solution of lacunary boundary value problem is founded by using B-spline of different degrees. This thesis is the starting point of this subject in the field of numerical analysis.

1.2 Interpolation [16]

Interpolation is a method used in numerical analysis to approximate functions or to estimate the value of a function $f(x)$ for arguments between $x_0, x_1, \dots, x_n$ at which the values $y_0, y_1, \dots, y_n$ are known.

The goal of this method is to replace a given function (whose values are known at determined points) by another one which is simpler. Interpolation has many applications: we know its values at specific points, approximating the integral and derivatives of function, and numerical solutions of integral and differential equation. The most used functions in interpolation are polynomials, trigonometric, exponentials and rational.

1.3 Lacunary Interpolation [16]

Lacunary interpolation appears whenever observation gives scattered or irregular information about a function and its derivatives. Also, the data in the problem of lacunary interpolation are values of the function and of its derivatives but without Hermite conditions that only consecutive derivative is used at each node. Mathematically, in the problem of interpolating a given data $a_{i,j}$ by a polynomial $P_n(x)$ of degree at most n satisfying:

$$P_n^{(j)}(x_i)\ , i = 1,2, \dots , n;\ j = 0,1,2, \dots , n \qquad (1.1)$$

We have Hermite interpolation if for each i, the order j of derivatives in Eq.1.1 form unbroken Sequence. If some of the sequences are broken, we have lacunary interpolation.

Polynomials are the most common choice of interpolations because they are easy to evaluate, differentiate and integrate

Higher order polynomials are not preferred because it is expected that the error between the function g and the polynomial approximation P^n on n sites to decrease when n increases. If the sites are uniformly spaced, it can be shown that this is not true and the interpolation error increases with n for some examples.

1.4 Spline [3]

For an interval $[a, b]$ is subdivided into sufficiently small intervals $[x_i, x_{i+1}]$, with $a = x_0 < \cdots < x_n = b$, on each such interval, a polynomial p_i of relatively low degree can provide a good approximation to g, This can even be done in such a way that the polynomial pieces blend smoothly, so that the resulting composite function $S(x)$ that equals $p_i(x)$ for $x \in [x_i, x_{i+1}]$, $0 \leq j \leq n - 1$, has several continuous derivatives. Any such smooth piecewise polynomial function is called a spline.

1.5 B-Spline [1]

A B-spline is a piecewise polynomial function of degree k in variable x. It is defined over a range $t_1 \leq x \leq t_m$, $m = k + 2$. The points where $x = t_j$ are known as knots or break-points. The knots must be in ascending order. The number of knots is the minimum for the degree of the B-spline, which has a non-zero value only in the range between the first and last knot. Each piece of the function is a polynomial of degree k between and including adjacent knots.

1.6 Properties of B-Spline [1]

A k^{th} degree B-Spline is denoted by $B_{i,k}(x), where\ i \in Z$, and it has the following properties, where $i = 0,1,2, \dots, n$

1) $B_{i,k}(x)$ is a non-zero polynomial on $[x_i, x_{i+k+1})$ for degree $k \geq 0$
2) On any span $[x_i, x_{i+1})$ at most $k + 1$ basis functions of degree k are non-zero, meaning
   $B_{i-k,k}(x), B_{i-k+1,k}(x), B_{i-k+2,k}(x), \dots, B_{i,k}(x)$ are non-zero
3) $\sum_{j=-\infty}^{j=\infty} c_j B_j(x) = 1$, $\forall x \in R$ (Partition of unity)
4) If x is outside the interval $[x_i, x_{i+k+1})$ then $B_{i,k}(x) = 0$ (support property)
5) $\frac{d}{dx} B_{i,k}(x) = \frac{k}{x_{i+k} - x_i} B_{i,k-1}(x) - \frac{k}{x_{i+k+1} - x_{i+1}} B_{i+1,k-1}(x)$
6) B-Spline has minimal support with respect to given degree, smoothness and domain partition,
7) B-spline is continuous at the knots. When all knots are distinct. Its derivatives are also continuous up to the derivative of degree $k - 1$. If knots are coincident at a given value of x, the continuity of derivative order is reduced by 1 for each additional knot.

8) For any given set of knots, the B-spline is unique, hence the name, B being short for Basis. The usefulness of B-spline lies in the fact that any spline function of degree k on a given set of knots can be expressed as a linear combination of B-spline as follows.

$$S(x) = \sum_{j=i-k+2}^{i+k-2} c_j B_j(x) \quad , \quad i = 0,1,2, \dots , n \tag{1.2}$$

1.7 Derivation of B-spline functions [1]

The B-splines were so called because it forms a basis for the set of all splines. Suppose that an infinite set of knots $\{x_i\}, i = 0,1,2, \dots , n$ is prescribed in a way that

$$\dots < x_{-2} < x_{-1} < x_0 < x_1 < x_2 < \cdots$$

The B-spline is depending on this set of knots.

Suppose a function f is defined as the set of points x when $f(x) \neq 0$

1.7.1 B-Splines of Degree Zero [1]

For $k = 0$, the B-Spline function is just a step function. the zero degree is one of the simplest B-Spline basis function and is given as

$$B_{i,0} = \begin{cases} 1, & x_i \leq x \leq x_{i+1} \\ 0, & Otherwise \end{cases} \quad , i = 0,1,2, \dots , n \tag{1.3}$$

1.7.2 B-Splines of Degree One [1]

The expression for the first degree B-spline, also called as linear B-spline can be obtained using the **Cox** and **De Boor** recursion formula given by Eq.1.4 in below;

$$B_{i,k}(x) = V_{i,k} B_{i,k-1}(x) + \left(1 - V_{i+1,k}\right) B_{i+1,k-1}(x) \tag{1.4}$$

Where $V_{i,k} = \frac{x - x_i}{x_{i+k} - x_i}$, and $i = 0,1,2, \dots , n$

Putting $k = 1$ in Eq.1.4 and use the definition of zero degree B-spline. The formula of the first degree B-spline is given as;

$$B_{i,1}(x) = \begin{cases} \frac{x - x_i}{x_{i+1} - x_i}, & x_i \le x < x_{i+1} \\ \frac{x_{i+2} - x}{x_{i+2} - x_{i+1}}, & x_{i+1} \le x < x_{i+2} \\ 0\,, & Otherwise \end{cases} \tag{1.5}$$

The first degree B-Spline is like a HAT which is non-zero for two knot spans $[x_i, x_{i+1}) and\ [x_{i+1}, x_{i+2})$, Where $i = 0,1,2, \dots, n$

1.7.3 B-Spline of Degree Two (Quadratic) [1]

Quadratic B-Spline can be obtained using the formula of linear B-spline basis function of Eq.1.5 and **Fox** and **De Boor** formula of Eq.1.4 for$k = 2$, the formula for quadratic B-spline is as follows where $i = 0,1,2, \dots, n$

$$B_{i,2}(x) = \begin{cases} \frac{(x - x_i)^2}{(x_{i-2} - x_i)(x_{i+1} - x_i)}, & x_i \le x < x_{i+1} \\ \frac{(x - x_i)(x_{i+2} - x)}{(x_{i+2} - x_i)(x_{i+2} - x_{i+1})} + \frac{(x_{i+2} - x)(x - x_{i+1})}{(x_{i+3} - x_{i+1})(x_{i+2} - x_{i+1})}, & x_{i+1} \le x < x_{i+2} \\ \frac{(x_{i+3} - x)^2}{(x_{i+3} - x_{i+1})(x_{i+3} - x_{i+2})}, & x_{i+2} \le x < x_{i+3} \\ 0\,, & Otherwise \end{cases} \tag{1.6}$$

1.7.4 B-Splines of degree three (Cubic B-spline) [1]

The third degree B-spline called as cubic B-spline is given by the following formula;

$$B_{i,3}(x) = \frac{1}{h^3} \begin{cases} (x - x_{i-2})^3, & x_{i-2} \le x < x_{i-1} \\ (x - x_{i-2})^3 - 4(x - x_{i-1})^3, & x_{i-1} \le x < x_i \\ (x_{i+2} - x)^3 - 4(x_{i+1} - x)^3, & x_i \le x < x_{i+1} \\ (x_{i+2} - x)^3, & x_{i+1} \le x < x_{i+2} \end{cases} \tag{1.7}$$

This definition of cubic B-spline basis function is given with x_i as the middle knot and equal number of knots on the two sides. Cubic B-spline is non-zero on four knot spans, the value of $B_{i,3}(x)$ can be obtained on the nodal points where $i = 0,1,2,\dots,n$

1.7.5 B-spline of degree four [1]

The B-spline basis function of fourth degree also called as Quartic B-Spline which can be derived from the recurrence formula of B-Spline, the formula will be given by;

$$B_{i,4}(x) = \frac{1}{h^4}\begin{cases} (x - x_{i-2})^4 & x_{i-2} \le x < x_{i-1} \\ (x - x_{i-2})^4 - 5(x - x_{i-1})^4 & x_{i-1} \le x < x_i \\ (x - x_{i-2})^4 - 5(x - x_{i-1})^4 + 10(x - x_i)^4 & x_i \le x < x_{i+1} \\ (x_{i+3} - x)^4 - 5(x_{i+2} - x)^4 & x_{i+1} \le x < x_{i+2} \\ (x_{i+3} - x)^4 & x_{i+2} \le x < x_{i+3} \\ 0 & Otherwise \end{cases} \tag{1.8}$$

This basis function is non-zero on five knots, the value of $B_{i,4}(x)$ at the nodal points can be obtained from Eq.1.9, Where $i = 0,1,2,\dots,n$

Putting k=4 in Equ1.2 , then the approximate solution will be as;

$$S(x) = \sum_{j=i-2}^{i+2} c_j B_j(x) \ , i = 0,1,\dots,n \tag{1.9}$$

In the next chapter, Quartic B-Spline is used to find an approximate solution of a boundary value problem

1.7.6 B-spline of degree five [1]

Also called Quintic B-spline, similar to the previous steps taken to find Cubic and Quartic B-spline, we can find the formula of Quintic B-spline which would be as follows;

$$B_{i,5}(x) = \frac{1}{h^5}\begin{cases} (x-x_{i-3})^5 & x_{i-3} \le x < x_{i-2} \\ (x-x_{i-3})^5 - 6(x-x_{i-2})^5 & x_{i-2} \le x < x_{i-1} \\ (x-x_{i-3})^5 - 6(x-x_{i-2})^5 + 15(x-x_{i-1})^5 & x_{i-1} \le x < x_i \\ (x_{i+3}-x)^5 - 6(x_{i+2}-x)^5 + 15(x_{i+1}-x)^5 & x_i \le x < x_{i+1} \\ (x_{i+3}-x)^5 - 6(x_{i+2}-x)^5 & x_{i+1} \le x < x_{i+2} \\ (x_{i+3}-x)^5, & x_{i+2} \le x < x_{i+3} \\ 0 & Otherwise \end{cases} \tag{1.10}$$

The basis function is non-zero on six knot spans, Where $i = 0,1,2,\dots,n$

1.7.7 B-spline of degree six [1]

Sextic B-spline can be obtained from the recurrence formula of B-spline and Quintic B-spline, the formula is given by;

$$S(x) = \begin{cases} (x_{i+5}-x)^6 - 7(x_{i+4}-x)^6 + 21(x_{i+3}-x)^6 - 35(x_{i+2}-x)^6 + 35(x_{i+1}-x)^6 - 21(x_i-x)^6 + 7(x_{i-1}-x)^6, & x_{i-2} \le x \le x_{i-1} \\ (x_{i+5}-x)^6 - 7(x_{i+4}-x)^6 + 21(x_{i+3}-x)^6 - 35(x_{i+2}-x)^6 + 35(x_{i+1}-x)^6 - 21(x_i-x)^6, & x_{i-1} \le x \le x_i \\ (x_{i+5}-x)^6 - 7(x_{i+4}-x)^6 + 21(x_{i+3}-x)^6 - 35(x_{i+2}-x)^6 + 35(x_{i+1}-x)^6, & x_i \le x \le x_{i+1} \\ (x_{i+5}-x)^6 - 7(x_{i+4}-x)^6 + 21(x_{i+3}-x)^6 - 35(x_{i+2}-x)^6, & x_{i+1} \le x \le x_{i+2} \\ (x_{i+5}-x)^6 - 7(x_{i+4}-x)^6 + 21(x_{i+3}-x)^6 & x_{i+2} \le x \le x_{i+3} \\ (x_{i+5}-x)^6 - 7(x_{i+4}-x)^6, & x_{i+3} \le x \le x_{i+4} \\ (x_{i+5}-x)^6, & x_{i+4} \le x \le x_{i+5} \\ 0, & otherwise \end{cases} \tag{1.11}$$

B-spline of degree 6 could be used find an approximate solution of a problem by substituting k=6 in Equ1.2, which gives;

$$S(x) = \sum_{j=i-4}^{i+4} c_j B_j(x) \quad , i = 0,1,2,\dots,n \tag{1.12}$$

Sextic B-Spline is used in chapter three to formulate an approximate solution of a boundary value problem.

Chapter Two

Lacunary Interpolation Using Quartic B-spline

2.1 Introduction

Boundary value problems of Ordinary differential equations, which is part of differential equation with conditions imposed at different points, has been applied in mathematics, engineering and various fields of sciences. The rapid increasing of its applications has led to formulating and upgrading several existed methods and new approaches [17]

Quartic B-spline is a piecewise polynomial of degree four satisfying third order parametric continuity. [16]

In this chapter, quartic B-spline is manipulated to approximate the solution of a BVP with lacunary data given on it and the boundary conditions are approximate. By presuming the B-spline to be the solution for this problem, an undetermined system of linear equations of order $(n+4)x(n+1)$ with n being the number of uniform subintervals is built. Adding three approximate boundary conditions into this system gives a square system of $(n+4)x(n+4)$ which is having a unique solution in this problem.

This method can make use of the problem's equation to construct an error equation. Minimization of the error equation would give the value of the variable that produces the best approximation of the solution.

2.2 lacunary Interpolations on Boundary Value Problem Using Quartic B-spline

The problem is to find an approximate solution of a function $f(x)$ having very limited lacunary data on it, the function has the following Interpolation conditions;

$$f(x_i) = y_i \qquad , i = 0,1,2,\dots,n \tag{2.1}$$

With the following lacunary boundary conditions

$f'(x_0) = y_0'$

$f'(x_n) = y_n'$ (2.2)

$f''(x_0) = y_0''$

The B-spline is non zero at five knots, we can find the value of $B_{i,4}$ at the nodal points by differentiating it with respect to x, the value of $B_{i,4}$ and its first three derivatives at the nodal points can be tabulated as in table 2.1

Table 2.1: Values of $B_{i,4}$ and its first three derivatives at the nodal points

	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}	x_{i+3}
$B_{i,4}(x)$	0	1	11	11	1	0
$B'_{i,4}(x)$	0	$4/h$	$12/h$	$-12/h$	$-4/h$	0
$B''_{i,4}(x)$	0	$12/h^2$	$-12/h^2$	$-12/h^2$	$12/h^2$	0
$B'''_{i,4}(x)$	0	$24/h^3$	$-72/h^3$	$72/h^3$	$-24/h^3$	0

From Eq.1.9, The solution of Eq.2.1 using Quartic B-spline is approximated by;

$$S(x) = \sum_{j=i-2}^{i+2} c_j B_j(x) \quad , \quad i = 0,1,2,\dots,n \tag{2.3}$$

then

$$S'(x) = \sum_{j=i-2}^{i+2} c_j B'_j(x) \quad , \quad i = 0,1,2,\dots,n \tag{2.4}$$

And

$$S''(x) = \sum_{j=i-2}^{i+2} c_j B''_j(x) \quad , \quad i = 0,1,2,\dots,n \tag{2.5}$$

Now without loss of generality, we can re-write Eq.2.3 as below,

$S(x) = c_{i-2}B_{i-2}(x_i) + c_{i-1}B_{i-1}(x_i) + c_iB_i(x_i) + c_{i+1}B_{i+1}(x_i) + c_{i+2}B_{i+2}(x_i)$

Where $i = 0,1,2,\dots,n$ and all other B_{i+k}'s are zero, $k = -4,-3,3,4$

By shifting the B spline to the right side by $k's$ step, mathematically meaning;

$B_{i-k}(x_i) = B_i(x_{i+k})$ for all .

then Eq.2.3 can be re-written as follows

$$S(x) = c_{i-2}B_i(x_{i+2}) + c_{i-1}B_i(x_{i+1}) + c_iB_i(x_i) + c_{i+1}B_i(x_{i-1}) + c_{i+2}B_i(x_{i-2}) \;, i = 0,1,2,\dots,n \tag{2.6}$$

Doing the same steps on first and second derivatives of the B-spline as in Eqs. 2.4 and 2.5 respectively in order to determine its value, gives

$$S'(x) = c_{i-2}B'_i(x_{i+2}) + c_{i-1}B'_i(x_{i+1}) + c_iB'_i(x_i) + c_{i+1}B'_i(x_{i-1}) + c_{i+2}B'_i(x_{i-2}) \;, i = 0,1,2,\dots,n \tag{2.7}$$

and

$$S''(x) = c_{i-2}B''_i(x_{i+2}) + c_{i-1}B''_i(x_{i+1}) + c_iB''_i(x_i) + c_{i+1}B''_i(x_{i-1}) + c_{i+2}B''_i(x_{i-2}) \;, i = 0,1,2,\dots,n \tag{2.8}$$

From the Lacunary conditions and on substituting the values of $B_{i,4}(x)$ at the knots from table 2.1, the following equations are formulated;

$$\begin{aligned} f(x_0) &\cong S(x_0) = c_{-2} + 11c_{-1} + 11c_0 + c_1 \\ f(x_1) &\cong S(x_1) = c_{-1} + 11c_0 + 11c_1 + c_2 \\ f(x_2) &\cong S(x_2) = c_0 + 11c_1 + 11c_2 + c_3 \\ &\;\;\vdots \\ f(x_n) &\cong S(x_n) = c_{n-2} + 11c_{n-1} + 11c_n + c_{n+1} \end{aligned} \tag{2.9}$$

From Eq.2.9, There are $n + 4$ unknowns to be founded and $n + 1$ equations, it is needed to write three more equations which are the lacunary boundary conditions, this gives;

$$hf'(x_0) \cong hS'(x_0) = -4c_{-2} - 12c_{-1} + 12c_0 + 4c_1$$

$$hf'(x_n) \cong hS'(x_n) = -4c_{n-2} - 12c_{n-1} + 12c_n + 4c_{n+1} \tag{2.10}$$

$$h^2 f''(x_0) \cong h^2 S''(x_0) = 12c_{-2} - 12c_{-1} - 12c_0 + 12c_1$$

Eqs. 2.9 and 2.10 forms a square system of $(n+4)x(n+4)$, the matrix form is as follows;

$$\begin{pmatrix}
1 & 11 & 11 & 1 & 0 & 0 & 0 & 0 & \cdots & & & 0 \\
-4 & -12 & 12 & 4 & 0 & 0 & 0 & 0 & \cdots & & & 0 \\
12 & -12 & -12 & 12 & 0 & 0 & 0 & 0 & \cdots & & & 0 \\
0 & 1 & 11 & 11 & 1 & 0 & 0 & 0 & \cdots & & & 0 \\
0 & 0 & 1 & 11 & 11 & 1 & 0 & 0 & & & & 0 \\
0 & 0 & 0 & 1 & 11 & 11 & 1 & 0 & \cdots & & & 0 \\
\cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \cdot \\
\cdot & \cdot & & & \cdot & \cdot & \cdot & \cdot & \cdot & & & \cdot \\
\cdot & \cdot & & & & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 11 & 11 & 1 & \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & -4 & -12 & 12 & 4 &
\end{pmatrix}_{(n+4)(n+4)}
\begin{pmatrix} c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \\ c_3 \\ \cdot \\ \cdot \\ \cdot \\ c_n \\ c_{n+1} \end{pmatrix}
=
\begin{pmatrix} f(x_0) \\ hf'(x_0) \\ h^2 f''(x_0) \\ f(x_1) \\ f(x_2) \\ f(x_3) \\ \cdot \\ \cdot \\ \cdot \\ f(x_n) \\ hf'(x_n) \end{pmatrix} \tag{2.11}$$

Above Matrix has a non-zero determinant which is concluded using MATLAB and n is taken arbitrarily, This yields that above system has a unique solution.

With this, the construction of a fourth degree B-spline is completed which is an approximate solution of the problem given by Eqs. 2.1 and 2.2.

2.3 Error bound

In this section, an error bound of the fourth degree B-spline that of section 2.1 is formulated

Denote Eq.2.11 by $AC = Y$. And

Let $\hat{S}(x) = \sum_{j=i-2}^{i+2} \hat{C}_j B_j(x)$ be an approximate solution of $\hat{Y}$, this is to be used in the coming lemmas.

Lemma 2.1

If A is an nxn matrix as defined in Eq.2.11, then $\|A^{-1}\|_\infty = \frac{1}{24}$

Proof:

First, it is a must to clarify that the matrix A has an inverse. In section 2.2, and using MATLAB, It is concluded that the square matrix A of Eq.2.11 has a non-zero determinant which means A is invertible. That is there exists an nxn matrix A^{-1} such that $AA^{-1} = I$, A^{-1} being the inverse of A.

It is known that $\|A\| = max\|Ax\|$, or $\|A\|_\infty = max_j \sum_{i=1}^{n} |aij|$, then

$$\|A^{-1}\| = max_{\|x\|=1}\|A^{-1}x\| = max_{\|Ay\|=1}\|y\| = \left(min_{\|Ay\|=1}\|y\|^{-1}\right)^{-1}$$

$$= (min\|Ax\|)^{-1}, \qquad where\ \ A^{-1}x = y$$

$$= (\min(24,32,48))^{-1} = \frac{1}{24}$$

Lemma 2.2

Let $\hat{S}(x) = \sum_{j=i-2}^{i+2} \hat{C}_j B_j(x)$ be another approximate solution of $\hat{Y}$ using exact boundary conditions, then

$$\|\hat{C} - C\|_\infty \leq \frac{1}{24} \|Y - \hat{Y}\|_\infty ,$$

Proof:

For the approximate solution $\hat{S}(x)$ of $\hat{Y}$, another matrix system could be obtained as follows;

$$A\hat{C} = \hat{Y} \text{ , and } AC = Y \rightarrow AC - A\hat{C} = Y - \hat{Y},$$

Then $A(C - \hat{C}) = (Y - \hat{Y}) \rightarrow (C - \hat{C}) = A^{-1}(Y - \hat{Y})$

Now using properties of norm, the following yields;

$$\|\hat{C} - C\|_\infty = \|A^{-1}(Y - \hat{Y})\|_\infty \leq \|A^{-1}\|_\infty \|Y - \hat{Y}\|_\infty \leq \frac{1}{24} \|Y - \hat{Y}\|_\infty$$

Lemma 2.3

The following inequalities are true for $i = 0,1,2, \dots, n$

i) $\sum_{i=-2}^{n+1} |B_i(x)| = 24$,

ii) $\sum_{i=-2}^{n+1} |B_i'(x)| \leq \frac{32}{h}$,

iii) $\sum_{i=-2}^{n+1} |B_i''(x)| \leq \frac{48}{h^2}$, and

iv) $\sum_{i=-2}^{n+1} |B_i'''(x)| \leq \frac{192}{h^3}$,

Proof:

For $x \in [x_i, x_{i+1}]$, and from Table 2.1, for $i = 0,1,2,\dots,n$

i) $\sum_{i=-2}^{n+1} |B_i(x)| = \sum_{i=-2}^{n+1} B_i(x) = 1 + 11 + 11 + 1 = 24$

ii) $\sum_{i=-2}^{n+1} |B_i{}'(x)| \leq \frac{4}{h} + \frac{12}{h} + \frac{12}{h} + \frac{4}{h} \leq \frac{32}{h}$

Similarly

iii) $\sum_{i=-2}^{n+1} |B_i{}''(x)| \leq \frac{48}{h^2}$

iv) $\sum_{i=-2}^{n=i+1} |B_i{}'''(x)| \leq \frac{192}{h^3}$

Lemma 2.4

Let $S(x)$ be a quartic B-spline approximate solution of $y(x)$ with lacunary and approximate boundary conditions, and let Ŝ(x) be another approximate solution of $y(x)$ with boundary conditions, then the followings are true for $i = 0,1,2,\dots,n$

i) $\left|S(x) - \hat{S}(x)\right| \leq \frac{1}{24} \left\|Y - \hat{Y}\right\|_\infty$

ii) $\left|S'(x) - \hat{S}'(x)\right| \leq \frac{4}{3h} \left\|Y - \hat{Y}\right\|_\infty$

iii) $\left|S''(x) - \hat{S}''(x)\right| \leq \frac{2}{h^2} \left\|Y - \hat{Y}\right\|_\infty$

iv) $\left|S'''(x) - \hat{S}'''(x)\right| \leq \frac{8}{h^3} \left\|Y - \hat{Y}\right\|_\infty$

Proof:

Using Lemma 2.2 and Lemma 2.3, and for $k = 0,1,2,3$, gives;

$$\left|S^{(k)}(x) - \hat{S}^{(k)}(x)\right| = \left|\sum_{i=-2}^{n+1} (C_i - \hat{C}_i) B_i^{(k)}(x)\right| \leq \left\|C - \hat{C}\right\|_\infty \sum_{i=-2}^{n+1} \left|B_i^{(k)}(x)\right|$$

i) For $k = 0$,

$$\left|S(x) - \hat{S}(x)\right| \leq \left\|C - \hat{C}\right\|_{\infty} \sum_{i=-2}^{i=n+1} |B_i(x)| \leq \left\|Y - \hat{Y}\right\|_{\infty}$$

ii) For $k = 1$,

$$\left|S'(x) - \hat{S}'(x)\right| \leq \left\|C - \hat{C}\right\|_{\infty} \sum_{i=-2}^{n+1} |B_i'(x)| \leq \frac{4}{3h}\left\|Y - \hat{Y}\right\|_{\infty}$$

iii) For $k = 2$;

$$\left|S''(x) - \hat{S}''(x)\right| \leq \left\|C - \hat{C}\right\|_{\infty} \sum_{i=-2}^{i=n+1} \left|B_i''(x)\right| \leq \frac{2}{h^2}\left\|Y - \hat{Y}\right\|_{\infty}$$

iv) Finally for $k = 3$;

$$\left|S'''(x) - \hat{S}'''(x)\right| \leq \left\|C - \hat{C}\right\|_{\infty} \sum_{i=-2}^{n+1} \left|B_i'''(x)\right| \leq \frac{8}{h^3}\left\|Y - \hat{Y}\right\|_{\infty}$$

In the subsequent section, we need the following values: For $f \in C^4[0,1]$, we have the following expansions.

$$\hat{S}(x_i) = y(x_i) + 2hy'(x_i) + 2h^2 y''(x_i) + \frac{4}{3}h^3 y_i'''(x_i) + \frac{2}{3}h^4 y_i^{(4)}(\theta_{1,i})$$

$$\hat{S}'(x_i) = y'(x_i) + 2hy''(x_i) + \frac{2}{3}h^2 y_i'''(x_i) + \frac{2}{9}h^3 y_i^{(4)}(\theta_{2,i})$$

$$\hat{S}''(x_i) = y''(x_i) + \frac{2}{3}h y_i'''(x_i) + \frac{2}{9}h^2 y_i^{(4)}(\theta_{3,i}) \tag{2.12}$$

$$\hat{S}'''(x_i) = y'''(x_i) + \frac{2}{3}h y_i^{(4)}(\theta_{4,i})$$

Where $\theta_{j,i} \in [x_0, x_n], j = 1,2,3,4$

Theorem 2.1

Let $S(x)$ be a 4th degree B-Spline approximation solution of Y, then the following inequalities are true;

i) $|S(x) - Y(\mathrm{x})| \leq \|Y - \hat{Y}\|_{\infty} + 2h\|y'(x_i)\| + 2h^2\|y_i''(x_i)\| + \frac{4}{3}h^3\|y_i'''(x_i)\| + w(y_i^{(4)}(\theta_{1,i}))$

ii) $|S'(x) - Y'(\mathrm{x})| \leq \frac{4}{3h}\ \|Y - \hat{Y}\|_{\infty} + 2h\|y''(x_i)\| + \frac{2}{3}h^2\|y_i'''(x_i)\| + \frac{2}{9}h^3 w(y_i^{(4)}(\theta_{2,i}))$

iii) $|S''(x) - Y''(\mathrm{x})| \leq \frac{2}{h^2}\ \|Y - \hat{Y}\|_{\infty} + \frac{2}{3}h\|y_i'''(x_i)\| + \ \frac{2}{9}h^2\ w\left(y^{(4)}(\theta_{3,i})\right)$

iv) $|S'''(x) - Y'''(\mathrm{x})| \leq \frac{8}{h^3}\ \|Y - \hat{Y}\|_{\infty} + \frac{2}{3}h\ w(f^{(4)}(\theta_{3,i}))$

Proof:

Using lemma 2.4, prove of (iv) is as follows;

$$|S'''(x) - Y'''(\mathrm{x})| \leq \left|S'''(x) - \hat{S}'''(\mathrm{x})\right| + \left|\hat{S}'''(x) - Y'''(\mathrm{x})\right|$$
$$\leq \frac{8}{h^3}\ \|Y - \hat{Y}\|_{\infty} + \left|\hat{S}'''(x) - Y'''(\mathrm{x})\right|$$

Assigning $\hat{S}'''(x)$ as $\hat{S}'''_i(x)$ and $Y'''(\mathrm{x})$ as $y_i'''(x_i)$,and using Eq.2.12, then the last term of above equation can be obtained as follows;

$$\left|\hat{S}'''(x_i) - Y'''(\mathrm{x_i})\right| = \left|y'''(x_i) + \frac{2}{3}hy_i^{(4)}(\theta_{1,i}) - y'''(x_i)\right| = \left|\frac{2}{3}hy^{(4)}(\theta_{4,i})\right|$$

Hence

$$|S'''(x) - Y'''(\mathrm{x})| \leq \frac{8}{h^3}\ \|Y - \hat{Y}\|_{\infty} + \frac{2}{3}h\ w\left(y^{(4)}(\theta_{4,i})\right), \text{where } i = 0,1,2,\dots,n$$

Similar to the proof of (iv) the followings can be proved;

iii) $|S''(x) - Y''(\mathrm{x})| \leq \left|S''(x) - \hat{S}''(\mathrm{x})\right| + \left|\hat{S}''(x) - Y''(\mathrm{x})\right| \leq \frac{2}{h^2}\ \|Y - \hat{Y}\|_{\infty} + \left|\hat{S}''(x) - Y''(\mathrm{x})\right|$

Assigning $\hat{S}''(x)$ as $\hat{S}''_i(x)$ and $Y''(\mathrm{x})$ as $y_i''(x_i)$, then the last term of above equation can be obtained as follows;

$$\left|\hat{S}''^{(x_i)} - Y''(\mathrm{x_i})\right| = \left|y''(x_i) + \frac{2}{3}hy_i'''(x_i) + \frac{2}{9}h^2y_i^{(4)}(\theta_{3,i}) - y''(x_i)\right|$$

$$= \left|\tfrac{2}{3} h y_i'''(x_i) + \tfrac{2}{9} h^2 y_i^{(4)}\left(\theta_{3,i}\right)\right|$$

$$= \frac{2}{3} h \| y_i'''(x_i) \| + \frac{2}{9} h^2 \, w\left(y^{(4)}\left(\theta_{3,i}\right)\right)$$

Hence

$$|S''(x) - Y''(\mathrm{x})| \leq \tfrac{2}{h^2} \, \left\|Y - \hat{Y}\right\|_\infty + \tfrac{2}{3} h \| y'''(x_i) \|_\infty + \tfrac{2}{9} h^2 \, w\left(y^{(4)}\left(\theta_{3,i}\right)\right),$$ where

$i = 0,1,2, \dots, n$

Applying similar techniques completes proof of the theorem

ii) $|S'(x) - Y'(\mathrm{x})| \leq \left|S'(x) - \hat{\mathrm{S}}'(x)\right| + \left|\hat{\mathrm{S}}'(x) - Y'(\mathrm{x})\right|$

$$\leq \tfrac{4}{3h} \, \left\|Y - \hat{Y}\right\|_\infty + \left|\hat{\mathrm{S}}'(x) - Y'(x)\right|$$

$$\leq \tfrac{4}{3h} \, \left\|Y - \hat{Y}\right\|_\infty + \left|\hat{\mathrm{S}}'(x_i) - Y'(x_i)\right|$$

$$\leq \frac{4}{3h} \, \left\|Y - \hat{Y}\right\|_\infty + \left|y'(x_i) + 2hy''(x_i) + \frac{2}{3} h^2 y_i'''(x_i) + \frac{2}{9} h^3 y_i^{(4)}(\theta_{2,i}) - y'(x_i)\right|$$

$$\leq \frac{4}{3h} \, \left\|Y - \hat{Y}\right\|_\infty + \left|2hy''(x_i) + \frac{2}{3} h^2 y_i'''(x_i) + \frac{2}{9} h^3 y_i^{(4)}(\theta_{2,i})\right|$$

$$\leq \frac{4}{3h} \, \left\|Y - \hat{Y}\right\|_\infty + 2h\|y''(x_i)\| + \frac{2}{3} h^2 \|y_i'''(x_i)\| + \frac{2}{9} h^3 w(y_i^{(4)}\left(\theta_{2,i}\right))$$

i) $|S(x) - Y(\mathrm{x})| \leq \left|S(x) - \hat{\mathrm{S}}(\mathrm{x})\right| + \left|\hat{\mathrm{S}}(x) - Y(\mathrm{x})\right|$

$$\leq \left\|Y - \hat{Y}\right\|_\infty + \left|\hat{\mathrm{S}}(x_i) - Y(x_i)\right|$$

$$\leq \left\|Y - \hat{Y}\right\|_\infty + \Big| y(x_i) + 2hy'(x_i) + 2h^2 y''(x_i) + \tfrac{4}{3} h^3 y_i'''(x_i) +$$

$$\tfrac{2}{3} h^4 y_i^{(4)}(\theta_{1,i}) - y(x_i) \Big| \; \leq \left\|Y - \hat{Y}\right\|_\infty + 2h\|y'(x_i)\| + 2h^2 \|y_i''(x_i)\| +$$

$$\tfrac{4}{3} h^3 \|y_i'''(x_i)\| + w(y_i^{(4)}\left(\theta_{1,i}\right))$$

Numerical Example 2.1

$y^{(4)}(x) + xy = -(8 + 7x + x^3)e^x \;\; with$

$y'(0) = 1$

$y'(1) = -1$

$y''(0) = 0$

The analytical solution is given by $y(x) = x(1 - x)e^x$

Table 2.2 compares the numerical results between present B-Spline method and the computational method used in [18]

Table 2.2

x_i	Exact Solution	[18]	B-Spline Max Error
1/8	-0.1239381121	$2.37e^{-8}$	$8.94e^{-9}$
1/16	-0.0437870146	$5.75e^{-9}$	$3.46e^{-9}$
1/32	-0.0312344196	$1.47e^{-9}$	$5.59e^{-10}$

Error of the present B-spline method is slightly better than the error obtained in [18], this confirms that the B-Spline method is a precise one for the models of BVPs stated in section 2.2

Numerical Example 2.2

$y^{(4)}(x) + 4y(x) = 1$,

$$y'(-1) = y'(1) = \frac{sinh2 - sin2}{4(cosh2 + cos2)}$$

$y''(0) = 1$

The results of maximum absolute error $\max\left[y^{(r)}(x_i)\right] = max_{1\le i\le n}\left|y^{(r)}(x_i) - S^{(r)}(x_i)\right|, r = 0,1,2,3$ are tabulated in Table 2.3

Table 2.3

x_i	Exact Solution	[18]	B-spline Error
1/8	0.07372	$1.29e^{-7}$	$7.53e^{-8}$
1/16	0.08767	$3.08e^{-8}$	$4.21e^{-9}$
1/32	0.08858	$7.54e^{-9}$	$3.17e^{-9}$

The example has been solved by Yogesh and Punkja. [18], The numerical results shown in Table 2.3 shows encouraging results of our method.

Chapter Three

Lacunary Interpolation
Using
Sextic B-spline

3.1 Introduction

Sextic B-spline is a piecewise polynomial of degree four satisfying fifth order parametric continuity.

In this chapter, Sextic B-spline is used to approximate the solution of a BVP with lacunary data and the boundary conditions are approximately given. By presuming the B-spline to be the solution for this problem, an undetermined system of linear equations of order $(n+6)x(n+1)$ with n being the number of uniform subintervals is built. Adding the five approximate boundary conditions given into this system gives a square system of $(n+6)x(n+6)$ which is must be a unique solution in this chapter.

In this method, an error equation is formulated. Minimization of the error equation would give the value of the variable that produces the best approximation of the solution.

3.2 Lacunary Interpolations on Boundary value problem Using Sextic B-spline

In this section time, B-spline of degree six is investigated to find the approximate solution of a boundary value problem which interpolates the function $f(x)$ at the function itself and has lacunary boundary conditions.

The interpolation condition is;

$$f(x_i) = y_i \qquad ,i = 0,1,2,\dots,n \tag{3.1}$$

And the lacunary boundary conditions are;

$$f'(x_0) = y_0^{'}$$

$$f'(x_n) = y_n^{'}$$

$$f''(x_0) = y_0^{''} \tag{3.2}$$

$f''(x_n) = y_n''$

$f'''(x_0) = y_0'''$

From Eq.1.11, as B-spline of degree six is given, we conclude that B-spline of degree six is non zero at seven knots, values of $B_{i,6}$ at the nodal points could be found through differentiation with respect to x, the following table 3.1 explains the values of $B_{i,6}$

Table 3.1: Values of $B_{i,6}$ and its first five derivatives at the nodal points.

	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}	x_{i+3}	x_{i+4}	x_{i-2}
$B_{i,6}(x)$	0	1	57	302	302	57	1	0
$B_{i,6}'(x)$	0	$6/h$	$150/h$	$240/h$	$-240/h$	$-150/h$	$-6/h$	0
$B_{i,6}''(x)$	0	$30/h^2$	$270/h^2$	$-300/h^2$	$-300/h^2$	$270/h^2$	$30/h^2$	0
$B_{i,6}'''(x)$	0	$120/h^3$	$120/h^3$	$-960/h^3$	$960/h^3$	$-120/h^3$	$-120/h^3$	0
$B_{i,6}^{(4)}(x)$	0	$360/h^4$	$-1080/h^4$	$720/h^4$	$720/h^4$	$-1080/h^4$	$360/h^4$	0
$B_{i,6}^{(5)}(x)$	0	$720/h^5$	$-3600/h^5$	$7200/h^5$	$-7200/h^5$	$3600/h^5$	$-720/h^5$	0

From Eq.1.11, The approximate solution using Sextic B-spline is given by;

$$S(x) = \sum_{j=i-4}^{i+4} c_j B_j(x) \tag{3.3}$$

then

$$S'(x) = \sum_{j=i-4}^{i+4} c_j B_j{}'(x) \tag{3.4}$$

$$S''(x) = \sum_{j=i-4}^{i+4} c_j B''_j(x) \tag{3.5}$$

Now we can re-write Eq.3.3 as in below,

$$S(x) = c_{i-4}B_{i-4}(x_i) + c_{i-3}B_{i-3}(x_i) + c_{i-2}B_{i-2}(x_i) + c_{i-1}B_{i-1}(x_i) + c_iB_i(x_i) + c_{i+1}B_{i+1}(x_i) + c_{i+2}B_{i+2}(x_i) + c_{i+3}B_{i+3}(x_i) + c_{i+4}B_{i+4}(x_i) \tag{3.6}$$

all other B_{i+k}'s are zero

Now, We shift the B_{i+k}'s to the right side by k's step, meaning

$B_{i-k}(x_i) = B_i(x_{i+k})$, then we can rewrite Eq..3.6 as follows;

$$S(x) = c_{i-4}B_i(x_{i+4}) + c_{i-3}B_i(x_{i+3}) + c_iB_{i-2}(x_{i+2}) + c_{i-1}B_i(x_{i+1}) + c_iB_i(x_i) + c_{i+1}B_i(x_{i-1}) + c_{i+2}B_i(x_{i-2}) + c_{i+3}B_i(x_{i-3}) + c_{i+4}B_i(x_{i-4}) \tag{3.7}$$

So

$$S'(x) = c_{i-4}B_i{}'(x_{i+4}) + c_{i-3}B_i{}'(x_{i+3}) + c_iB_{i-2}{}'(x_{i+2}) + c_{i-1}B_i{}'(x_{i+1}) + c_iB_i{}'(x_i) + c_{i+1}B_i{}'(x_{i-1}) + c_{i+2}B_i{}'(x_{i-2}) + c_{i+3}B_i{}'(x_{i-3}) + c_{i+4}B_i{}'(x_{i-4}) \quad (3.8)$$

$$S''(x) = c_{i-4}B_i{}''(x_{i+4}) + c_{i-3}B_i{}''(x_{i+3}) + c_iB_{i-2}{}''(x_{i+2}) + c_{i-1}B_i{}''(x_{i+1}) + c_iB_i{}''(x_i) + c_{i+1}B_i{}''(x_{i-1}) + c_{i+2}B_i{}''(x_{i-2}) + c_{i+3}B_i{}''(x_{i-3}) + c_{i+4}B_i{}''(x_{i-4}) \quad (3.9)$$

$$S'''(x) = c_{i-4}B_i{}'''(x_{i+4}) + c_{i-3}B_i{}'''(x_{i+3}) + c_iB_i{}'''(x_{i+2}) + c_{i-1}B_i{}'''(x_{i+1}) + c_iB_i{}'''(x_i) + c_{i+1}B_i{}'''(x_{i-1}) + c_{i+2}B_i{}'''(x_{i-2}) + c_{i+3}B_i{}'''(x_{i-3}) + c_{i+4}B_i{}'''(x_{i-4}) \quad (3.10)$$

From Eqs.3.7-3.10 and using interpolation condition from Eq.3.1 and on substituting the values of $B_{i,6}(x)$ at the knots from table 2.1, then the following equations are formulated;

$$f(x_0) = c_{-4} + 57c_{-3} + 302c_{-2} + 302c_{-1} + 57c_0 + 57c_1$$

$$f(x_1) = c_{-3} + 57c_{-2} + 302c_{-1} + 302c_0 + 57c_1 + 57c_2$$

$$f(x_2) = c_{-2} + 57c_{-1} + 302c_0 + 302c_1 + 57c_2 + 57c_3$$

$$\vdots \qquad (3.11)$$

$$f(x_n) = c_{n-4} + 57c_{n-3} + 302c_{n-2} + 302c_{n-1} + 57c_n + 57c_{n+1}$$

In Eq.3.11, There are $n+6$ unknowns to be found and $n+1$ equations, it is needed to obtain five more equations which are the lacunary boundary conditions of Eq.3.2 and as below;

$$hf'(x_0) = -6c_{-4} - 150c_{-3} - 240c_{-2} + 240c_{-1} + 150c_0 + 6c_1$$

$$hf'(x_n) = -6c_{n-4} - 150c_{n-3} - 240c_{n-2} + 240c_{n-1} + 150c_n + 6c_{n+1}$$

$$h^2 f''(x_0) = 30c_{-4} + 270c_{-3} - 300c_{-2} - 300c_{-1} + 270c_0 + 30c_1 \tag{3.12}$$

$$h^2 f''(x_n) = 30c_{n-4} + 270c_{n-3} - 300c_{n-2} - 300c_{n-1} + 270c_n + 30c_{n+1}$$

$$h^3 f'''(x_0) = -120c_{-4} - 120c_{-3} + 960c_{-2} - 960c_{-1} + 120c_0 + 120c_1$$

Eq.3.11 and Eq.3.12 forms an $(n+6)x(n+6)$ square system, the matrix form is as follows;

$$\begin{pmatrix}
1 & 57 & 302 & 302 & 57 & 1 & 0 & 0 & \cdots & & & 0 \\
-6 & -150 & -240 & 240 & 150 & 6 & 0 & 0 & \cdots & & & 0 \\
30 & 270 & -300 & -300 & 270 & 30 & 0 & 0 & \cdots & & & 0 \\
-120 & -120 & 960 & -960 & 120 & 120 & 0 & 0 & \cdots & & & 0 \\
0 & 1 & 57 & 302 & 302 & 57 & 1 & 0 & & & & 0 \\
0 & 0 & 1 & 57 & 302 & 302 & 57 & 1 & \cdots & & & 0 \\
0 & 0 & 0 & 1 & 57 & 302 & 302 & 57 & 1 & & & 0 \\
. & . & & . & . & . & . & . & . & & & . \\
. & . & & & . & . & . & . & . & . & & . \\
. & . & & & & . & . & . & . & . & & . \\
0 & 0 & \cdots & 0 & 0 & 1 & 57 & 302 & 302 & 57 & & 1 \\
0 & 0 & \cdots & 0 & 0 & -6 & -150 & -240 & 240 & 150 & & 6 \\
0 & 0 & \cdots & 0 & 0 & 30 & 270 & -300 & -300 & 270 & & 30
\end{pmatrix}_{(n+6)(n+6)}
\begin{pmatrix} c_{-4} \\ c_{-3} \\ c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \\ . \\ . \\ . \\ c_{n-1} \\ c_n \\ c_{n+1} \end{pmatrix}
=
\begin{pmatrix} f(x_0) \\ hf'(x_0) \\ h^2 f''(x_0) \\ h^3 f'''(x_0) \\ f(x_1) \\ f(x_2) \\ f(x_3) \\ . \\ . \\ . \\ f(x_n) \\ hf'(x_n) \\ h^2 f''(x_n) \end{pmatrix} \tag{3.13}$$

To find c_i's, n can arbitrarily be taken. Using MATLAB it is concluded that above Matrix has a non-zero determinant which means the system in Eq.3.10 has a unique solution. This completes the construction of the sextic B-spline, $S(x)$ is an approximate solution of the lacunary boundary value problem given in Eqs.3.1 and 3.2.

3.3 Error bound

In this section, the error bound of the sixth degree B-spline will be founded that we constructed in section (3.1).

Rewrite Eq.3.13 as $AC = Y$ and

Let $\hat{S}(x) = \sum_{j=i-4}^{i+4} \hat{C}_j B_j(x)$ be an approximate solution of $\hat{Y}$, this is to be used in the coming lemmas.

Lemma 3.1

If A is an nxn matrix as defined in Eq.3.13, then $\|A^{-1}\|_\infty = \frac{1}{720}$

Proof:

The proof is similar to the technique used in the proof of Lemma 2.1. First, it is a must to clarify that the matrix A has an inverse. From section 3.2, and using MATLAB, It is concluded that the square matrix A of Eq.3.13 has a non-zero determinant which means A is invertible. That is there exists an nxn matrix A^{-1} such that $AA^{-1} = I$, A^{-1} being the inverse of A.

It is known that $\|A\| = max\|Ax\|$, or $\|A\|_\infty = max_j \sum_{i=1}^{n} |aij|$, then

$$\|A^{-1}\| = max_{\|x\|=1}\|A^{-1}x\| = max_{\|Ay\|=1}\|y\| = \left(min_{\|Ay\|=1}\|y\|^{-1}\right)^{-1}$$

$$= (min\|Ax\|)^{-1}\,, \qquad where\ \ A^{-1}x = y$$

$$= (\min(720,792,1200,2400))^{-1} = \frac{1}{720}$$

Lemma 3.2

The following inequality holds

$$\left\|\hat{C}-C\right\|_\infty \le \frac{1}{720}\left\|\hat{Y}-Y\right\|_\infty$$

Proof:

For the approximate solution $\hat{S}(x)$ of $\hat{Y}$, another matrix system could be obtained as follows;

$A\hat{C}=\hat{Y}$, and

$AC=Y \rightarrow AC-A\hat{C}=Y-\hat{Y},$

Then $A\left(C-\hat{C}\right)=\left(Y-\hat{Y}\right) \rightarrow \left(C-\hat{C}\right)=A^{-1}\left(Y-\hat{Y}\right)$

Now using properties of norm, the following yields;

$$\left\|\hat{C}-C\right\|_\infty=\left\|A^{-1}(Y-\hat{Y})\right\|_\infty \le \|A^{-1}\|_\infty\left\|Y-\hat{Y}\right\|_\infty \le \frac{1}{720}\left\|Y-\hat{Y}\right\|_\infty$$

Lemma 3.3

The followings are true

i) $\sum_{i=-4}^{n+1}|B_i(x)|=720,$

ii) $\sum_{i=-4}^{n+1}|B_i'(x)| \le \frac{792}{h}$

iii) $\sum_{i=-4}^{n+1}|B_i''(x)| \le \frac{1200}{h^2}$

iv) $\sum_{i=-4}^{n+1}|B_i'''(x)| \le \frac{2400}{h^3}$

v) $\sum_{i=-4}^{n+1}\left|B_i^{(4)}(x)\right| \le \frac{4320}{h^4}$ and

vi) $\sum_{i=-4}^{n+1}\left|B_i^{(5)}(x)\right| \le \frac{23040}{h^5}$

Proof

For $x \in [x_i, x_{i+1}]$, and from Table 3.1, the following can be obtained;

i) $\sum_{i=-4}^{n+1} |B_i(x)| = 1 + 57 + 302 + 302 + 57 + 1 = 720$

The rest inequalities can be proved similar to the proof of (i),

ii) $\sum_{i=-4}^{n+1} |B_i{}'(x)| = \sum_{i=-4}^{n+1} B_i{}'(x) \leq \frac{792}{h}$

iii) $\sum_{i=-4}^{n+1} |B_i{}''(x)| = \sum_{i=-4}^{n+1} B_i{}''(x) \leq \frac{1200}{h^2}$

iv) $\sum_{i=-4}^{n+1} |B_i{}'''(x)| = \sum_{i=-4}^{n+1} B_i{}'''(x) \leq \frac{2400}{h^3}$

v) $\sum_{i=-4}^{n+1} \left|B_i{}^{(4)}(x)\right| = \sum_{i=-4}^{n+1} B_i{}^{(4)}(x) \leq \frac{4320}{h^4}$

vi) $\sum_{i=-4}^{n+1} \left|B_i{}^{(5)}(x)\right| = \sum_{i=-4}^{n+1} B_i{}^{(5)}(x) \leq \frac{23040}{h^5}$

Lemma 3.4

Let $S(x)$ be a B-spline of degree six and an approximate solution of $y(x)$ with interpolation and lacunary boundary conditions, and let $\hat{S}(x)$ be another Approximate solution of $y(x)$ with interpolation and exact boundary conditions, then the following inequalities hold;

i) $\left|S(x) - \hat{S}(x)\right| \leq \left\|Y - \hat{Y}\right\|_\infty$

ii) $\left|S'(x) - \hat{S}'(x)\right| \leq \frac{11}{10h} \left\|Y - \hat{Y}\right\|_\infty$

iii) $\left|S''(x) - \hat{S}''(x)\right| \leq \frac{5}{3h^2} \left\|Y - \hat{Y}\right\|_\infty$

iv) $\left|S'''(x) - \hat{S}'''(x)\right| \leq \frac{10}{3h^3} \left\|Y - \hat{Y}\right\|_\infty$

v) $\left|S^{(4)}(x) - \hat{S}^{(4)}(x)\right| \leq \frac{6}{h^4} \left\|Y - \hat{Y}\right\|_\infty$

vi) $\left|S^{(5)}(x) - \hat{S}^{(5)}(x)\right| \leq \frac{32}{h^5} \left\|Y - \hat{Y}\right\|_\infty$

Proof:

Using Lemmas 3.2 and 3.3, and considering $k = 0,1,2,3,4,5$, gives;

$$\left|S^{(k)}(x) - \hat{S}^{(k)}(x)\right| = \left|\sum_{i=-4}^{n+1}(C_i - \hat{C}_i)B_i^{(k)}(x)\right| \le \left\|C - \hat{C}\right\|_\infty \sum_{i=-4}^{n+1}\left|B_i^{(k)}(x)\right|$$

i) For $k = 0$

$$\left|S(x) - \hat{S}(x)\right| \le \left\|C - \hat{C}\right\|_\infty \sum_{i=-4}^{i=n+1}|B_i(x)| \le \left\|Y - \hat{Y}\right\|_\infty$$

ii) For $k = 1$

$$\left|S'(x) - \hat{S}'(x)\right| \le \left\|C - \hat{C}\right\|_\infty \sum_{i=-4}^{i=n+1}|B_i{}'(x)| \le \frac{11}{10h}\left\|Y - \hat{Y}\right\|_\infty$$

iii) For $k = 2$

$$\left|S''(x) - \hat{S}''(x)\right| \le \left\|C - \hat{C}\right\|_\infty \sum_{i=-4}^{i=n+1}|B_i{}''(x)| \le \frac{5}{3h^2}\left\|Y - \hat{Y}\right\|_\infty$$

iv) For $k = 3$;

$$\left|S'''(x) - \hat{S}'''(x)\right| \le \left\|C - \hat{C}\right\|_\infty \sum_{i=-4}^{i=n+1}|B_i{}'''(x)| \le \frac{10}{3h^3}\left\|Y - \hat{Y}\right\|_\infty$$

v) For $k = 4$

$$\left|S^{(4)}(x) - \hat{S}^{(4)}(x)\right| \le \left\|C - \hat{C}\right\|_\infty \sum_{i=-4}^{i=n+1}\left|B_i{}^{(4)}(x)\right| \le \frac{6}{h^4}\left\|Y - \hat{Y}\right\|_\infty$$

vi) For $k = 5$

$$\left|S^{(5)}(x) - \hat{S}^{(5)}(x)\right| \le \left\|C - \hat{C}\right\|_\infty \sum_{i=-4}^{i=n+1}\left|B_i{}^{(5)}(x)\right| \le \frac{32}{h^5}\left\|Y - \hat{Y}\right\|_\infty$$

In the subsequent section, we need the following values: For $f \in C^6[0,1]$, we have the following expansions;

$$\hat{S}(x_i) = y(x_i) + 2hy'(x_i) + 2h^2y''(x_i) + \frac{4}{3}h^3y_i'''(x_i) + \frac{2}{3}h^4y_i^{(4)}(x_i) + \frac{4}{15}h^5y_i^{(5)}(x_i) + \frac{4}{45}h^6y_i^{(6)}(\theta_{1,i})$$

$$\hat{S}'(x_i) = y'(x_i) + \frac{1}{2}y''(x_i) + 2hy_i'''(x_i) + 2h^2y_i^{(4)}(x_i) + \frac{4}{3}h^3y_i^{(5)}(x_i) + \frac{2}{3}h^4y_i^{(6)}(\theta_{2,i}) \tag{3.14}$$

$$\hat{S}''(x_i) = y''(x_i) + \frac{2}{3}hy_i'''(x_i) + \frac{2}{9}h^2y_i^{(4)}(x_i) + \frac{4}{81}h^3y_i^{(5)}(x_i) + \frac{2}{243}h^4y_i^{(6)}(\theta_{3,i})$$

$$\hat{S}'''(x_i) = y'''(x_i) + \frac{2}{3}hy_i^{(4)}(x_i) + \frac{2}{9}h^2y_i^{(5)}(x_i) + \frac{4}{81}h^3y_i^{(6)}(\theta_{4,i})$$

$$\hat{S}^{(4)}(x_i) = y_i^{(4)}(x_i) + 2hy_i^{(5)}(x_i) + 2h^2y_i^{(6)}(\theta_{5,i})$$

$$\hat{S}^{(5)}(x_i) = y_i^{(5)}(x_i) + \frac{2}{3}hy_i^{(6)}(\theta_{6,i})$$

Where $\theta_{j,i} \in [x_0, x_n], j = 1,2,3,4,5,6$

Theorem 3.1

Let $S(x)$ be the Sextic B-spline which is an approximate solution of Y, then the following inequalities are true;

i) $|S(x) - Y(x)| \le \|Y - \hat{Y}\|_\infty +$

ii) $|S'(x) - Y'(x)| \le \frac{11}{10h}\|Y - \hat{Y}\|_\infty +$

iii) $|S''(x) - Y''(x)| \le \frac{5}{3h^2}\|Y - \hat{Y}\|_\infty + \frac{2}{3}h\|y_i'''(x_i)\| + \frac{2}{9}h^2\left\|y_i^{(4)}(x_i)\right\| + \frac{4}{81}h^3\left\|y_i^{(5)}(x_i)\right\| + \frac{2}{243}h^4\left\|y_i^{(6)}(\theta_{3,i})\right\|$

iv) $|S'''(x) - Y'''(x)| \le \frac{10}{3h^3}\|Y - \hat{Y}\|_\infty + \frac{2}{3}h\left\|y_i^{(4)}(x_i)\right\| + \frac{2}{9}h^2\left\|y_i^{(5)}(x_i)\right\| + \frac{4}{81}h^3w(y_i^{(6)}(\theta_{4,i}))$

v) $\left|S^{(4)}(x)-Y^{(4)}(x)\right| \leq \frac{6}{h^4}\,\|Y-\hat{Y}\|_\infty + 2h\|y^{(5)}(x_i)\| + 2h^2 w\left(y^{(6)}(\theta_{5,i})\right)$

vi) $\left|S^{(5)}(x)-Y^{(5)}(x)\right| \leq \frac{32}{h^5}\,\|Y-\hat{Y}\|_\infty + \frac{2}{3}h\,w\left(y^{(6)}(\theta_{6,i})\right)$

Proof:

Using lemma 2.4, prove of (vi) is as follows;

$$\left|S^{(5)}(x)-Y^{(5)}(x)\right| \leq \left|S^{(5)}(x)-\hat{S}^{(5)}(x)\right| + \left|\hat{S}'''(x)-Y^{(5)}(x)\right|$$
$$\leq \frac{32}{h^5}\|Y-\hat{Y}\|_\infty + \left|\hat{S}^{(5)}(x)-Y^{(5)}(x)\right|$$

Assigning $\hat{S}^{(5)}(x)$ as $\hat{S}^{(5)}{}_i(x)$ and $Y^{(5)}(x)$ as $y_i^{(5)}(x_i)$, and using Eq.3.14, then the last term of above equation can be obtained as follows;

$$\left|\hat{S}^{(5)}{}_i(x)-\ Y_i^{(5)}(x_i)\right| = \left|y^{(5)}(x_i)+\frac{2}{3}hy_i^{(6)}(\theta_{6,i})-y^{(5)}(x_i)\right| = \left|\frac{2}{3}hy^{(6)}(\theta_{6,i})\right|$$

Hence

$$\left|S^{(5)}(x)-Y^{(5)}(x)\right| \leq \frac{32}{h^5}\|Y-\hat{Y}\|_\infty + \frac{2}{3}h\,w\left(y^{(6)}(\theta_{6,i})\right), \text{where } i=0,1,2,\dots,n$$

Proof of (vi) is completed

Similarly (v) can be proved;

$$\left|S^{(4)}(x)-Y^{(4)}(x)\right| \leq \left|S^{(4)}(x)-\hat{S}^{(4)}(x)\right| + \left|\hat{S}^{(4)}(x)-Y^{(4)}(x)\right|$$
$$\leq \frac{6}{h^4}\,\|\hat{Y}-Y\|_\infty + \left|\hat{S}^{(4)}(x)-Y^{(4)}(x)\right|$$

Assigning $\hat{S}^{(4)}(x)$ as $\hat{S}^{(4)}{}_i(x)$ and $Y^{(4)}(x)$ as $y_i^{(4)}(x_i)$,and using Eq.3.14, then the last term of above equation can be obtained as follows;

$$\left|\hat{S}^{(4)}{}_i(x)-\ Y_i^{(4)}(x_i)\right| = \left|y^{(4)}(x_i)+2hy_i^{(5)}(x_i)+2h^2y^{(6)}(\theta_{5,i})-\right.$$
$$\left.y^{(4)}(x_i)\right| = 2h\|y^{(5)}(x_i)\| + 2h^2w\left(y^{(6)}(\theta_{5,i})\right)$$

Hence

$\left|S^{(4)}(x)-Y^{(4)}(x)\right| \le \frac{6}{h^4}\left\|Y-\hat{Y}\right\|_\infty + 2h\left\|y^{(5)}(x_i)\right\| + 2h^2 w\left(y^{(6)}\left(\theta_{5,i}\right)\right)$, where $i = 0,1,2,\dots,n$

Proof of (v) is completed.

In the same way, we prove the remaining of the theorem;

iv) $\left|S'''^{(x)} - Y'''(x)\right| \le \left|S'''^{(x)} - \hat{S}'''(x)\right| + \left|\hat{S}'''^{(x)} - Y'''(x)\right| \le \frac{10}{3h^3}\left\|Y-\hat{Y}\right\|_\infty + \left|y'''(x_i) + \frac{2}{3}hy_i^{(4)}(x_i) + \frac{2}{9}h^2y_i^{(5)}(x_i) + \frac{4}{81}h^3y_i^{(6)}\left(\theta_{4,i}\right) - y'''(x_i)\right| \le \frac{10}{3h^3}\left\|Y-\hat{Y}\right\|_\infty + \frac{2}{3}h\left\|y_i^{(4)}(x_i)\right\| + \frac{2}{9}h^2\left\|y_i^{(5)}(x_i)\right\| + \frac{4}{81}h^3 w(y_i^{(6)}\left(\theta_{4,i}\right))$

iii) $|S''(x) - Y''(\mathrm{x})| \le \left|S''(x) - \hat{S}''(x)\right| + \left|\hat{S}''(x) - Y''(x)\right| \le \frac{5}{3h^2}\left\|Y-\hat{Y}\right\|_\infty + \left|y''(x_i) + \frac{2}{3}hy_i'''(x_i) + \frac{2}{9}h^2y_i^{(4)}(x_i) + \frac{4}{81}h^3y_i^{(5)}(x_i) + \frac{2}{243}h^4y_i^{(6)}\left(\theta_{3,i}\right) - y''(x_i)\right| \le$

$\frac{5}{3h^2}\left\|Y-\hat{Y}\right\|_\infty + \left|\frac{2}{3}hy_i'''(x_i) + \frac{2}{9}h^2y_i^{(4)}(x_i) + \frac{4}{81}h^3y_i^{(5)}(x_i) + \frac{2}{243}h^4y_i^{(6)}\left(\theta_{3,i}\right)\right| \le \frac{5}{3h^2}\left\|Y-\hat{Y}\right\|_\infty + \frac{2}{3}h\|y_i'''(x_i)\| + \frac{2}{9}h^2\left\|y_i^{(4)}(x_i)\right\| + \frac{4}{81}h^3\left\|y_i^{(5)}(x_i)\right\| + \frac{2}{243}h^4 w\left(y_i^{(6)}\left(\theta_{3,i}\right)\right)$

ii) $|S'(x) - Y'(x)| \le \left|S'(x) - \hat{S}'(x)\right| + \left|\hat{S}'(x) - Y'(x)\right| \le \frac{11}{10\mathrm{h}}\left\|Y-\hat{Y}\right\|_\infty + \left|y'(x_i) + \frac{1}{2}y''(x_i) + 2hy_i'''(x_i) + 2h^2y_i^{(4)}(x_i) + \frac{4}{3}h^3y_i^{(5)}(x_i) + \frac{2}{3}h^4y_i^{(6)}\left(\theta_{2,i}\right) - y'(x_i)\right| \le \frac{11}{10\mathrm{h}}\left\|Y-\hat{Y}\right\|_\infty + \frac{1}{2}\|y''(x_i)\| + 2h\|y_i'''(x_i)\| + 2h^2\left\|y_i^{(4)}(x_i)\right\| + \frac{4}{3}h^3\left\|y_i^{(5)}(x_i)\right\| + \frac{2}{3}h^4 w\left(y_i^{(6)}\left(\theta_{2,i}\right)\right)$

i) $|S(x)-Y(x)| \leq \left|S(x)-\hat{S}(x)\right| + \left|\hat{S}(x)-Y(x)\right| \leq \left\|Y-\hat{Y}\right\|_\infty + \Big|y(x_i)+2hy'(x_i)+2h^2y''(x_i)+\frac{4}{3}h^3y_i'''(x_i)+\frac{2}{3}h^4y_i^{(4)}(x_i)+\frac{4}{15}h^5y_i^{(5)}(x_i)+\frac{4}{45}h^6y_i^{(6)}(\theta_{1,i})-y(x_i)\Big| \leq \left\|Y-\hat{Y}\right\|_\infty + \Big|\frac{4}{3}h^3y_i'''(x_i)+\frac{2}{3}h^4y_i^{(4)}(x_i)+\frac{4}{15}h^5y_i^{(5)}(x_i)+\frac{4}{45}h^6y_i^{(6)}(\theta_{1,i})\Big| \leq \left\|Y-\hat{Y}\right\|_\infty + \frac{4}{3}h^3\|y'''(x_i)\| + \frac{2}{3}h^4\left\|y_i^{(4)}(x_i)\right\| + \frac{4}{15}h^5\left\|y_i^{(5)}(x_i)\right\| + \frac{4}{45}h^6 + \frac{4}{45}h^6w\,(y_i^{(6)}(\theta_{1,i}))$

Numerical Example 3.1

Consider the BVP

$y^{(6)}(x)+y(x)=6cosx\,, \qquad 0\leq x\leq 1$

$y'(0)=0$

$y'(1)=0$

$y''(0)=1$

$y''(1)=0$

$y'''(0)=-1$

The exact solution is given by $y(x)=(x-1)sinx$

Table 3.2

x_i	Exact Solution	[12]	B-Spline Error
1/10	-0.0015707956	$3.2599e^{-6}$	$7.5092e^{-7}$
1/20	-0.0008290313	$1.3846e^{-7}$	$2.2619e^{-7}$
1/40	-0.0004254240	$2.8847e^{-8}$	$1.3017e^{-7}$
1/80	-0.0002154391	$1.3493e^{-8}$	$1.8632e^{-8}$

The results of maximum absolute error $\max\left[y^{(r)}(x_i)\right] = max_{1\leq i\leq n}\left|y^{(r)}(x_i) - S^{(r)}(x_i)\right|, r = 0,1,2,3$ for this problem are tabulated in Table 3.2

Numerical Example 3.2

Consider the BVP

$y^{(6)}(x) + y(x) = -6e^{x}$,

$y'(0) = 1$

$y'(1) = 0$

$y''(0) = 1$

$y''(1) = 0$

$y'''(0) = -1$

The exact solution is given by $y(x) = (1 - x)e^{x}$

The result of maximum absolute error is

$$\max\left[y^{(r)}(x_i)\right] = max_{1 \le i \le n}\left|y^{(r)}(x_i) - S^{(r)}(x_i)\right|, r = 0,1,2,3,4,5$$

The example has been solved using B-spline of degree six and the numerical results are stated in Table 3.3 below;

Table 3.3

x_i	Exact Solution	[12]	B-spline Error
0.1	0.99465383	$4.5092e^{-6}$	$2.9614e^{-6}$
0.2	0.97712221	$1.2619e^{-5}$	$2.7581e^{-6}$
0.3	0.94490117	$1.9154e^{-5}$	$1.8243e^{-5}$
0.4	0.89509482	$2.1632e^{-5}$	$1.9631e^{-5}$
0.5	0.82436064	$1.9704e^{-5}$	$1.8573e^{-5}$
0.6	0.72884752	$1.4548e^{-5}$	$6.3297e^{-6}$
0.7	0.60412581	$8.2238e^{-6}$	$3.0134e^{-6}$

The example has been solved by Li, M.; Chen, L. and Ma, Q. [12], which shows a comparision with other methods as well which are used to solve this example. The numerical results shown in Table 3.3 shows encouraging results of our method.

Chapter Four

Conclusion
and
Future Works

4.1 Conclusion

Based on the investigation done in the present study, it can be concluded that, B-Spline is fast, flexible and precise to be used to find approximate solutions of boundary value problems with given limited lacunary interpolation condition and approximate boundary condition, below are some conclusions;

1) B- Spline produced an approximation of analytical solution of the problem with respect to the selected subinterval.
2) B-Spline is a good tool to be used to solve Lacunary interpolation problems for Boundary Value Problems
3) It is considerable that as the subintervals are increasing, and *h* being small, the approximation by B-Spline is more precise and has potential to give good approximation solution for boundary value problems
4) B-Spline of degree six is a good approximation solution to BVP models as in chapter three.
5) B-Spline of degree four is precise and flexible enough to become an approximate solution of BVPs.
6) The errors in the numerical examples in chapter two and chapter three are quite good, slightly better than other the errors obtained from other approximation tools.

4.2 Future Works

It is recommended to conduct a research on one of the following topics;

1) A similar study on lacunary interpolation to find approximate solution of initial value problems
2) Another lacunary interpolation with other degrees can be used to solve initial and boundary value problems

3) Lacunary Interpolation using B-spline to find approximate solution of higher order boundary value problems and the result to be compared with other numerical methods

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الخلاصة

كرست هذه الدراسة لآيجاد بي- سبلاين يندرج بيانات فراغية معطاة عن دالة معينة مع وجود شروط حدودية تقريبية حول الدالة , البي – سبلاين ستكون حلا تقريبيا للدالة .

في البداية نقوم بتعريف الأندراج باستخدام بي – سبلاين , انواع ال(بي- سبلاين) , درجاتها و خواصها , بعدها نبدء ببحث حالات معينة من الأندراج الفراغي باستخدام البي – سبلاين .

المهم في الفصل الأول هي معادلة بي-سبلاين من الدرجة الرابعة و السادسة حيث سنقوم باستخدامها في الفصلين الثاني و الثالث.

في الفصل الثاني , نقوم ببناء ب – سبلاين من الدرجة الرابعة كحلِ تقريبي لدالة معينة معطياة بياناتها و شروطها الحدودية محدودوة جدا و تقريبية , يليها ايجاد الخطأ التقريبي لمتعددة البي – سبلاين التي اوجدناها.

في الفصل الثالث , ندرس و نقوم ببناء ب-سبلاين من الدرجة السادسة و ايجادها كحل تقريبي لدالة محدودة الشروط و معطياة البيانات عليها عشوائية و محدودة لكنها مختلفة عن الدالة المذكورة في الفصل الثاني. الشروط الحدودية للدوال التي نجد حلولها التقريبية هي شروط تقريبية و بيانات الأندراج هي فراغية و محدودة جدا.

ايجاد معاملات البي-سبلاين ستكون من خلال بناء معادلات معينة من المعادلة الرئيسية للبي-سبلاين و هذا من خلال شرط الأندراج , يليها بناء معادلات من الشروط الحدويوة التقريبية و الحصول على عدد من المعادلات عددها تساوي عدد المعاملات المجهولة في معادلة البي-سبلاين.

للوصول الى نتيجة سريعة و دقيقة, استخدمنا برنامج الماتلاب لأيجاد حلول المصفوفات المعقدة التي نحصل عليها جراء بناء البي – سبلاين في الفصلين الثاني و الثالث , من خلال الماتلاب نستنتج و نتأكد في ما اذا كانت المعادلات او نظام المصفوفات لديها حلول وحيدة او لا.

دراسة

بعض حالات

الأندراج الفراغي بأستخدام ال بي– سبلاين

رسالة مقدمة الى

مجلس كلية العلوم في جامعة السليمانية

كجزء من متطلبات نيل شهادة الماجستر في علم الرياضيات

(التحليل العددي)

من قبل

بژار جمال عزيز

بكالوريوس في الرياضيات (٢٠٠٦) , جامعة السليمانية

بأشراف

د. كاروان حمه فرج جوامير

بروفيسور

شوال ١٤٣٧   آب ٢٠١٦

پوختە

ئەم نامەیە بۆ دۆزینەوەی (بی– سپلاین)ە کە گونجاو بێت بۆ شیکاری پڕکردنەوەی بۆشاییەکانی فەنکشنێك کە چەند مەرجێکی باوندەری نزیککراوەیی لەسەر دراوە , ئەم بی – سپلاینە ئەبێتە حەلی نزیککراوەیی بۆ فەنکشنەکە.

سەرەتا پێناسەی ئینتەرپۆلەیشن بە بەکارهێنانی بی – سپلاین ئەکەین , جۆرەکانی بی – سپلاین , نمرەکانی و سیفەتەکانی , دواتر دەست ئەکەین بە لێکۆڵینەوە لە ئینتەرپۆلەیشن بە بەکارهێنانی بی – سپلاین .

ئەوەی گرنگە لە بەشی یەکەمدا هاوکێشەی بی – سپلاینی نمرە چوار و شەشە چونکە ئەمانەمان بەکارهێناوە لە بەشی دووەم و سێیەم.

لە بەشی دووەم, بی– سپلاینی پلە چوار دروست ئەکەین و ئەیکەینە حەلی نزیککراوەیی بو فەنکشنەکە کە چەند زانیاریەکی وۆر سنووورداری لەسەر دراوە لە مەرجی لاکونەری ئینتەرپۆلەیشن و مەرجی باوندەری نزیککراوە., پاشان هەڵەی نزیککراوەیی بی-سپلاینەکە ئەدۆزینەوە بۆ ئەوەی بزانین ڕاددەی هەڵەکە چەندە .

لە بەشی سێیەمدا, بی – سپلاینی پلە شەش دروست ئەکەین بۆ ئەوەی ببێتە حەلێکی نزیککراوە لە فەنکشنێك کە زانیاری لەسەری زۆر سنورداره و نا ڕێك و پێکە و تەنانەت مەرجە باوندەریەکانیشی کەمن و نزیککراوەن ,

مەرجە باوندەریەکانی ئەو فەنکشنەی ئەمانەوێت حەلی نزیککراوەیی بۆ بدۆزینەوە بە بەکارهێنانی بی – سپلاین , مەرجی نزیککراوەیین , وە مەرجی لاکونەری ئینتەرپۆلەیشنەکەش زۆر سنورداره .

دۆزینەوەی هاوکۆلکەکانی بی – سپلاینەکە لە ڕێی دروست کردنی چەند هاوکێشەیەکەوە ئەبێت لە هاوکێشەی سەرەکی بی – سپلاین ئەویش بە بەکارهێننی مەرجی لاکونەری ئینتەرپۆلەیشن . پاشان دروست کردنی چەند هاوکێشەیەکی تر لەڕێی مەرجە باوندەریە نزیککراوەییەکان بە جۆریك کە ژمارەی هاوکێشەکان یەکسان بێت بە ژمارەی نەزانراوەکان.

بۆ گەیشتن بە حەل بە خێرایی و بە ووردی , بەرنامەی ماتلاب بەکار ئەهێنین بو دۆزینەوەی حەلی ماتریکسەکان کە لە دروست کردنی بی-سپلاینەکەدا یەتە ڕێمان , لە ڕێگەی ماتلابەوە ئەتوانین بزانین و دڵنیا بین لەوەی کە سیستەمی چوارگۆشەیی هاوکێشەکە یان سیستمی ماتریکسەکە حەلی تەنهای هەیە یان نەو .

لێکۆڵینەوەی
هەندێك نموونەی
پڕکردنەوەی بۆشاییەکان
بە بەکارهێنانی بی– سپڵاین

نامەیەکە

پێشکەشکراوە بە

ئەنجومەنی کۆلێجی زانست لە زانکۆی سلێمانی

وەك بەشێك لە پێداویستیەکانی بەدەستهێنانی بڕوانامەی ماستەر لە زانستی ماتماتیك

(شیکاری ژمارەیی)

لە لایەن

بژار جمال عزیز

بەکالۆریۆس لە ماتماتیك (٢٠٠٦) , زانکۆی سلێمانی

بە سەرپەرشتی

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پرۆفیسۆر

خەرمانان ٢٧١٦ ئاب ٢٠١٦